SYZ mirror symmetry conjecture

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Outline

- Review on the SYZ program.
- Directions and current progress.
- Future perspectives.

Section 1

The SYZ program

Mirror symmetry

- Mirror symmetry was discovered in the early 90's by **Greene-Plesser** and **Candelas-De la Ossa-Green-Parkes**.

- It asserts that **Calabi-Yau manifolds come in pairs** $(X, \check{X})$. It gives the duality

$$\text{Complex geometry}(X) \longleftrightarrow \text{Symplectic geometry}(\check{X})$$

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- Countings of rational curves are reflected from period integrals of the mirror. This is known as **closed-string mirror symmetry**, which was proved by **Givental** and **Lian-Liu-Yau**.

- **Counting rational curves** was a very difficult task in classical algebraic geometry. For instance, the number of degree 3 curves in the Fermat quintic was computed by Kontsevich in 1994, which is 317206375. **Mirror symmetry gives the countings for all degrees simultaneously**.

- The BIG question is: **WHY does mirror symmetry occur**?

Calabi-Yau manifolds

Theorem (Calabi conjecture)

Let (X, ω_0) be a compact Kähler manifold with $c_1(X) = 0$. Then there exists a unique Kähler metric ω on X such that $\mathrm{Ric}(\omega) = 0$ and $[\omega] = [\omega_0]$.

- (X, ω) in the above theorem is called a **Calabi-Yau manifold**.
- String theory models our universe by $X \times \mathbf{R}^{3,1}$. To preserve supersymmetry, (X, ω) has to be Kähler with $\mathrm{Ric}(\omega) = 0$. Yau's theorem gives a complete solution to this equation.
- Locally

$$\mathcal{M}^{\text{CY metric}} \equiv \mathcal{M}^{\text{complex}} \times \mathcal{M}^{\text{Kähler}}.$$

  Thus we should consider both **complex geometry** and **symplectic geometry** to understand Calabi-Yau manifolds. Mirror symmetry asserts that these two geometries are exchanged in the mirror.

The SYZ conjecture

- In 1996, **Strominger**-**Yau**-**Zaslow** proposed that *mirror symmetry is T-duality*.

- Consider a point moving in a Calabi-Yau manifold $\check{X}$, which is a complex geometric object, calibrated in the sense of **Harvey**-**Lawson**. It should corresponds to a special Lagrangian submanifold L in the mirror X. As the point moves around the whole $\check{X}$, deformations of L should also fill up the whole X. In good cases it should give a **special Lagrangian fibration** on X.

- The deformation space of a point has $\dim_{\mathbf{C}} = 3$. Thus the deformation space of L, which is $h^1(L)$, should also be 3. This suggests that L is a three-torus, which has $h^1 = 3$ and has a natural dual.

- To complexify the deformation space, we should consider deformations of (L, ∇), where ∇ is a flat $U(1)$ connection over L. Then $\check{X}$ is the moduli space of such (L, ∇).

- If we fix L and only vary ∇, it gives the dual torus of L. Thus $\check{X}$ can be **reconstructed as the total space of the dual Lagrangian torus fibration**.

The SYZ conjecture

Physical arguments were used to make sense of the above simplified picture. Mathematically it could be formulated into the following.

Conjecture

Let X and $\check{X}$ be a mirror pair of Calabi-Yau manifolds near the large complex structure limits.

1. *X and $\check{X}$ admit **dual special Lagrangian torus fibrations** $\mu : X \to B$ and $\check{\mu} : \check{X} \to B$ over the same base B. Namely for a regular value $b \in B$, $\mu^{-1}(b)$ and $\check{\mu}^{-1}(b)$ are dual tori.*

2. *There exists a **fiberwise Fourier-Mukai transform** which maps Lagrangian submanifolds of X to coherent sheaves over $\check{X}$.*

Important conjectural consequences

- For a mirror pair of Calabi-Yau manifolds, conjecturally **they should always have special Lagrangian torus fibrations**, at least around the large complex structure limit.

- By the result **Hitchin**, the moduli space of special Lagrangians is unobstructed and has a natural affine structure.

- The fibration should gradually collapse to the base at the large volume limit. At the limit the torus fibers become flat, and the base should have an affine metric satisfying a real Monge-Ampere equation. This opens a way for **studying Calabi-Yau geometry by using tropical geometry of the base**.

- The mirror complex geometry can be reconstructed from taking dual torus fibrations. It gives an SYZ map

$$\mathbf{f}^{\mathrm{SYZ}} : \mathcal{M}^{\mathrm{Kähler}}(X) \to \mathcal{M}^{\mathrm{complex}}(\check{X}).$$

- **Fiberwise Fourier-Mukai transform provides a correspondence between the geometric objects in the two sides of the mirror.** It indicates that there should be a uniform geometric approach to derive **homological mirror symmetry**.

Fierwise Fourier-Mukai transform

$$\mathcal{P}$$

$$\downarrow$$

$$X \times_B \check{X}$$

$$\pi \swarrow \qquad \check{\pi} \searrow$$

$$X \qquad \qquad \check{X}$$

$$\searrow \qquad \swarrow$$

$$B$$
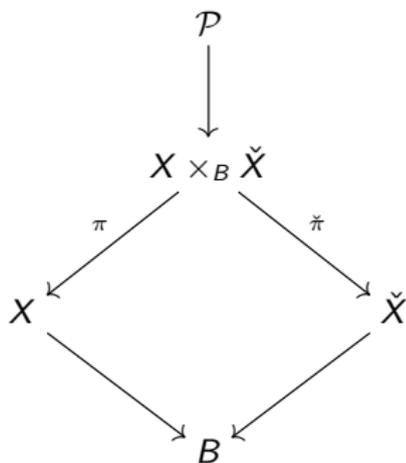
$$\mathcal{F}^{\mathrm{SYZ}}(-) := \check{\pi}_*(\pi^*(-) \otimes \mathcal{P}).$$

- $\mathcal{P}$ is the Poincaré line bundle, namely $\mathcal{P}|_{X \times_B \{\check{x}\}}$ is the flat connection over $X \times_B \{\check{x}\} \cong T$ parametrized by $\check{x}$.
- It transforms Lagrangian fibers of $X \to B$ to skyscraper sheaves over a point in $\check{X}$, and Lagrangian sections in X to holomorphic line bundles over $\check{X}$.
- **Leung-Yau-Zaslow** successfully constructed the transform for semi-flat branes.
- T-duality for toric manifolds was further developed by **Leung-Vafa**, which eventually lead to a physical proof of mirror symmetry by **Hori-Vafa** and **Iqbal-Hori-Vafa**.

Family Floer theory

- Family Floer theory was proposed by **Fukaya** to transform Lagrangian branes in X to sheaves over the SYZ mirror $\check{X}$.

- Given a Lagrangian brane L, one constructs the family of morphism spaces $\mathrm{CF}^*((T, \nabla), L)$ over $(T, \nabla) \in \check{X}$. This gives a sheaf $\mathcal{L}$ over $\check{X}$ corresponding to L.

- A morphism $p \in \mathrm{CF}^*(L_1, L_2)$ induces the morphism $m_2(p, -)$ from $\mathcal{L}_1$ to $\mathcal{L}_2$.

- This approach was pursued by **Tu** and **Abouzaid** independently. It gives an A_∞ functor from the Fukaya category of X to the derived category of coherent sheaves over the SYZ mirror $\check{X}$, at least when there are no singular fibers in the Lagrangian fibration.

Extension to non-Calabi-Yau geometries

- ▶ Mirror symmetry has well-known extensions beyond Calabi-Yau geometries. The mirrors are given by **Landau-Ginzburg models**, which are complex varieties $\check{X}$ equipped with holomorphic functions $W : \check{X} \to \mathbf{C}$.

- ▶ The SYZ program naturally extends to construct the Landau-Ginzburg mirrors. W is defined by counting of holomorphic discs bounded by Lagrangian torus fibers. **Auroux** studied wall-crossing and gave a general recipe to construct LG mirrors.

- ▶ Motivated by SYZ, **Hori-Vafa** gave a recipe to construct mirrors of toric manifolds. The SYZ construction was carried out and computed by **Cho-Oh** and **Chan-Leung** in the Fano case, which agree with the **Hori-Vafa** mirror.

- ▶ In general the toric SYZ mirrors were constructed by **Fukaya-Oh-Ohta-Ono**, which admit non-trivial quantum corrections due to complicated bubblings. For $-K_X \geq 0$, the corrections were fully computed by **Chan-Lau-Leung-Tseng**.

# Quantum corrections to Hori-Vafa mirrors
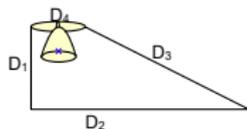
- ▶ Holomorphic discs bounded by Lagrangian torus fiber are the essential ingredients of the SYZ construction. Each disc contributes a monomial in the superpotential W.

- ▶ For a toric manifold, each toric prime divisor corresponds to one **basic holomorphic disc**. The Hori-Vafa mirror reads

$$W = \sum_i Z_{\beta_i}$$

where β_i are the basic discs and Z_{β_i} are the corresponding monomials.

- ▶ In the non-Fano case, there are **rational curves with Chern number** ≤ 0. They can be glued with several copies of the basic discs which contribute additional terms to W. This gives **quantum corrections to the Hori-Vafa mirror**.

Quantum corrections to Hori-Vafa mirrors
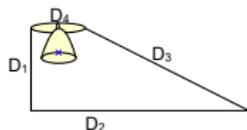
- Moduli spaces of the additional discs carry highly technical **obstructions**. Tools in closed Gromov-Witten theory cannot be applied, because the moduli spaces have **codimension-one boundaries**.

- They were studied extensively by **Auroux** for the Hirzebruch surfaces $\mathbf{F}_2$ and $\mathbf{F}_3$ using wall-crossing, and **Fukaya-Oh-Ohta-Ono** for $\mathbf{F}_2$ using degeneration techniques.

- The work of **Chan-Lau-Leung-Tseng** computed the full formulas for the superpotentials of all $(-K_X)$-nef toric manifolds in terms of the mirror maps. They identified the holomorphic discs in X with section curves of certain $\mathbf{P}^1$ bundles over X, and computed them using Seidel representations.

- For instance, for $X = \mathbf{P}(K_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2})$,

$$\sum_{k \geq 0} n_{\beta_0 + kl} q^{kl} = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + 35870q^6 - \dots$$

where β_0 denotes the basic disc class emanated from the zero section, and l denotes the line class of $\mathbf{P}^2$.

Key difficulties to overcome in the SYZ program

- **Constructing special Lagrangian fibrations** on compact Calabi-Yau manifolds is a very difficult task. For instance, it is not known if the quintic threefold admits a special Lagrangian fibration. Indeed finding one special Lagrangian in a given class is already a very difficult open problem in differential geometry.

- **Quantum corrections** occur to all orders for reconstructing the mirror complex structure. Intuitively they are holomorphic discs emanating from singular fibers, which interact with each other and scatter in a complicated way. Reconstructing the mirrors of compact Calabi-Yau manifolds poses a major challenge.

- Family Floer theory also receives **quantum corrections coming from singular fibers.** Formulating the transform in the presence of quantum corrections is the major difficulty in deriving homological mirror symmetry from SYZ.

Section 2

Directions and current progress

We will discuss and investigate the following directions.

- **Construction of special Lagrangian fibrations.**
- **SYZ mirror construction.**
- **Calabi-Yau metric around the large complex structure limit.**
- **SYZ transformation of branes.**
- **SYZ and the mirror maps.**
- **Twin Lagrangian fibrations.**

Constructing special Lagrangian fibrations

- **Goldstein** and **Gross** constructed Lagrangian fibrations for toric Calabi-Yau n-folds, which generalizes the fibrations of **Harvey**-**Lawson** on $\mathbf{C}^3$. They show that these fibrations are special with respect to certain holomorphic volume forms.

- The construction **uses a T^{n-1}-symmetry on the n-fold**. Namely, they observed that there is a natural T^{n-1} action on every toric Calabi-Yau n-fold X. The symplectic reduction $X//T^{n-1}$ can be identified with the complex plane.

- Then they show that the circle fibration $r = |z - 1|$ on the complex plane can be pulled back to give a Lagrangian fibration on the n-fold. The circle fibration is special with respect to the holomorphic one-form $d \log(z - 1)$. As a result, there exists a holomorphic volume form on the n-fold such that the Lagrangian fibration is special.

- Construction of Lagrangian fibrations is **much more difficult in the compact case** since there is generically no continuous symmetry to help.
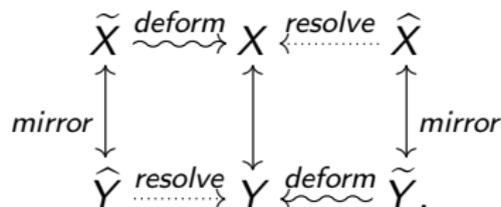
Constructing special Lagrangian fibrations

- **W.D. Ruan** asserted that a Lagrangian torus fibration on the quintic CY can be constructed using **Hamiltonian flow and toric degenerations**.

- At the limit of a Hamiltonian flow, the quintic CY degenerates to a union of several copies of $\mathbb{P}^3$. Each $\mathbb{P}^3$ has a Lagrangian fibration given by the moment map. Ruan analyzed the degeneration, and asserted that the **Lagrangian fibration at the limit can be pulled back to a Lagrangian fibration on the quintic along the flow.**

- **Gross** used a topological version of this fibration to derive topological mirror symmetry for the quintic, namely, the quintic and its mirror has topologically dual torus fibrations.

Constructing special Lagrangians

▶ Constructing a **special** Lagrangian submanifold is much more difficult than constructing a Lagrangian submanifold. It is the global volume minimizer in its homology class by the calibration theory of **Harvey-Lawson**.

▶ The main idea is to minimize a Lagrangian using **mean curvature flow**. This idea was pursued by **Thomas-Yau**. They found that **stability** is essential in this approach.

▶ Roughly speaking, an object L is **stable** if for every sub-objects L', the phase of the central charge of L' is smaller than that of L. The central charge of a Lagrangian is defined by its period $\int_L \Omega$.

▶ **Douglas** studied **BPS branes** in $(2, 2)$ superconformal field theory and formulated the notion of Π-**stability** which describes how the BPS branes vary over the moduli. **Bridgeland** made a rigorous mathematical formulation of the axioms using the language of triangulated category. In this context the BPS branes are special Lagrangians.

Conifold transitions

$$\widetilde{X} \xrightarrow{\text{deform}} X \xleftarrow{\text{resolve}} \widehat{X}$$

$$\big\uparrow \text{mirror} \qquad \big\uparrow \qquad \big\downarrow \text{mirror}$$

$$\widehat{Y} \xrightarrow{\text{resolve}} Y \xleftarrow{\text{deform}} \widetilde{Y}.$$

▶ It is a general principle by **Morrison** that the process of conifold transition get reversed under mirror symmetry as shown above.

▶ Note that **Kähler property in general are not preserved under conifold transition**. **Clemens-Friedman-Tian** constructed complex structure on $\widehat{X}$, which may not admit a Kähler form. **Smith-Thomas-Yau** constructed a symplectic structure on $\widetilde{Y}$, which may not admit a complex structure. They derived the **explicit obstructions to the existence of Kähler structures**.

▶ It is an interesting to ask how to use SYZ to understand conifold transition under mirror symmetry. **Castano-Bernard and Matessi** studied the change of Lagrangian fibrations under conifold transitions. However mirror symmetry for **non-Kähler structures** were not understood yet.

SYZ construction of mirror manifolds

- ► Quantum correction to T-duality is the key to reconstruct the mirror manifolds using SYZ approach. It comes from holomorphic discs emanating from singular Lagrangian fibers.

- ► **Auroux** gave an extensive study of the **wall-crossing phenomenon**. He discovered that the countings of holomorphic discs changes drastically when a Lagrangian torus fiber moves across certain walls in the base.

- ► Using the techniques of wall-crossing of holomorphic discs, **Chan-Lau-Leung** constructed the SYZ mirrors of all toric Calabi-Yau manifolds.

- ► **Abouzaid-Auroux-Katzarkov** constructed the SYZ mirrors of blow-ups of toric varieties associated to hypersurfaces. It gives the reverse direction of the construction of Chan-Lau-Leung.

- ► In general the configurations of walls are very complicated, due to **scattering of holomorphic discs**.

# SYZ construction of toric Calabi-Yau manifolds
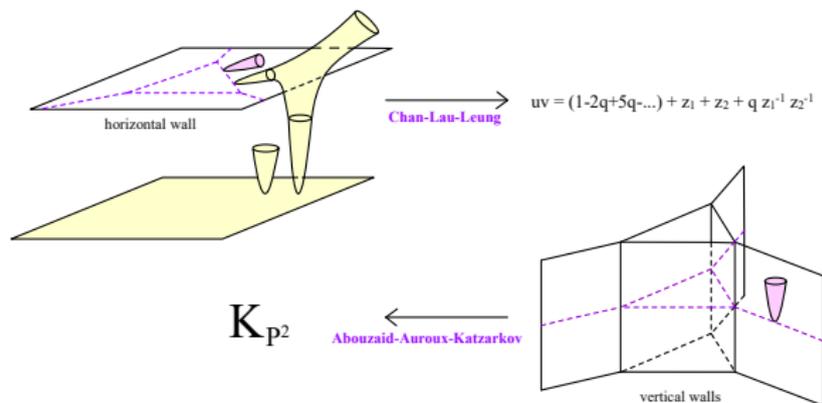
- **Wall-crossing of holomorphic discs studied by Auroux** is the key to understand SYZ construction.
- The figure shows $K_{\mathbf{P}^2}$ as an example. There is a codimension-one horizontal wall in the base of the Lagrangian fibration. Holomorphic discs over the wall interact with a holomorphic disc below the wall to produce more discs above the wall. **Chan-Lau-Leung** constructed its SYZ mirror by fully computing the disc countings.
- **Abouzaid-Auroux-Katzarkov** constructed the Lagrangian fibration on the mirror conic fibration. The discriminant locus is (homotopically) the same, but the walls become vertical. They analyzed the wall-crossing and recover $K_{\mathbf{P}^2}$ as the mirror.
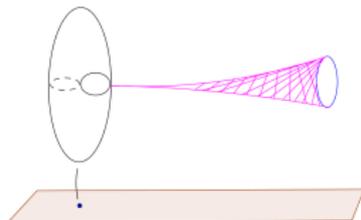
Quantum corrections and scattering

- ► The pioneering works of **Kontsevich** and **Fukaya** proposed that quantum corrections for reconstructing the mirror are **holomorphic discs emanating from singular fibers**. They interact with each other in a complicated way, which is difficult to study directly using symplectic technique.

- ► **Kontsevich and Soibelman** produced a universal **wall-crossing formula** in deformation theory, and asserted that in particular scatterings of holomorphic discs obey the formula.

- ► **Gross and Siebert** developed a **reconstruction program** of mirrors which uses the wall-crossing formula to define the quantum corrections combinatorially. This avoids the difficulty of constructing special Lagrangian fibrations and handling holomorphic discs.

- ► To connect with homological mirror symmetry, one needs to understand the *precise relations between their combinatorial approach and the geometric approach*.

Quantum corrections and scattering
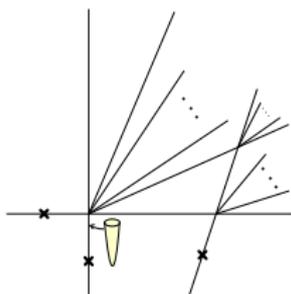
- ▶ The figure shows the base of a Lagrangian torus fibration.
- ▶ Each cross represents a singularity over which the fiber is singular.
- ▶ Rays emanating from crosses correspond to holomorphic discs bounded by torus fibers.

Quantum corrections and scattering

▶ Two disks emanating from different singular fibers can be glued with a pair of pants to give a new disk.

Interaction : scattering

scattered walls

B^{sing}
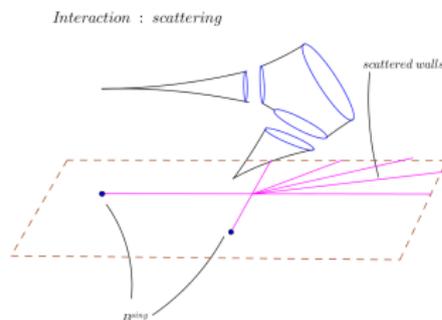
▶ Gluing of holomorphic disks should be recorded by a scattering diagram which is purely combinatorial. It was extensively studied by **Gross**-**Pandharipande**-**Siebert**.

▶ Each ray is attached with an automorphism on $(\mathbf{C}^{\times})^n$ which records the disc countings. The wall-crossing formula is a compatibility condition says that the ordered composition of automorphisms clockwise around intersection point of two rays is an identity.

Differential geometric approach to scattering

- In dimension 2, **Fukaya** suggested that the mirror complex manifold can be obtained from deformation of the semi-flat dual fibration by a 'distributional' solution ϕ, supported on the rays. The Maurer-Cartan equation is

$$\bar{\partial}\phi + \frac{1}{2}\{\phi, \phi\} = 0$$

- **Chan-Leung-Ma** developed a differential geometric method to reconstruct the scattering process. The scattering diagram results as a semi-classical limit of distributional solutions to the Maurer-Cartan equation. A smoothing parameter $\hbar > 0$ is introduced in the program, and the distributions are series in $\hbar$. The leading order term is a distribution supported on the scattering diagram.

- The method is explained in more detail in the following slide.

Differential geometric approach to scattering
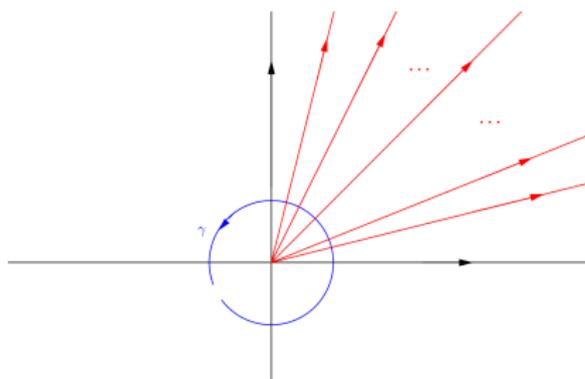
- Each incoming (black) ray corresponds to a smooth section of $\Omega^{0,1}(((\mathbf{C}^{\times})^n), T^{1,0})$. They give the initial data for deformation of the complex structure.
- When two incoming rays intersect, the sum of the two sections does not solve the Maurer-Cartan equation.
- By **Chan-Leung-Ma**, the equation can be solved order-by-order in $\hbar$ by using propagators. Taking the semi-classical limit $\hbar \to 0$, the solution is supported on the new graph with additional (red) rays.

Calabi-Yau metrics near large complex structure limit

- In the large complex structure limit, it is expected that the special Lagrangian fibration collapses to the base in the sense of metric, and the Calabi-Yau metric approaches to the semi-flat metric of the base. The true Calabi-Yau metric should be approximated by gluing local CY models around singular fibers with the semi-flat metric.

- For K3 surfaces, this approach of constructing Calabi-Yau metric was pursued by **Gross-Wilson**. However, more information is needed to compute all the instanton corrections.

- The semi-flat metric with singularities were constructed by **Greene-Shapere-Vafa-Yau**. The local CY model around each singular fiber is constructed by **Ooguri-Vafa**.

- In dimension three, the problem becomes much more difficult, since the discriminant loci of Lagrangian fibrations contain **'Y'-shapes**. **Loftin-Yau-Zaslow** extensively studied the affine geometry and constructed the semi-flat metric around a 'Y'-shape discriminant locus.

SYZ transformation of Lagrangian cycles

- ▶ SYZ transforms Lagrangian sections to holomorphic line bundles on the mirror side. **Leung-Yau-Zaslow** gave an extensive study of this transform for semi-flat branes.

- ▶ Using this approach, **Abouzaid** proved a version of the homological mirror symmetry conjecture for smooth projective toric varieties.

- ▶ **Fang-Liu-Treumann-Zaslow** formulated the transform in the language of constructible sheaves and Morse theory. They proved another version of homological mirror symmetry for smooth projective toric varieties.

- ▶ **Chan-Ueda** and **Chan-Pomerleano-Ueda** constructed Fourier-Mukai–type transforms (SYZ transforms) for 2- and 3-dimensional toric Calabi-Yau manifolds, where one encounters SYZ fibrations with singular fibers and hence nontrivial quantum corrections; It give equivalences between the derived category of coherent sheaves on the toric Calabi-Yau manifold and the wrapped Fukaya category on its mirror.

The deformed Hermitian-Yang-Mills equation

▶ In their study of SYZ transforms for **semi-flat special Lagrangian branes**, **Leung-Yau-Zaslow** considered the deformed Hermitian-Yang-Mills equation, which seeks a connection A on the mirror line bundle L with curvature $\omega = iF_A$ satisfying

$$Im\left(e^{-i\hat\theta}(\alpha + i\omega)^n\right) = 0$$

where α is the background Kähler form and $\hat\theta$ is a fixed Lagrangian angle.

▶ **Mariño-Minasian-Moore-Strominger** first formulated this equation when looking at open-string effective actions and studying BPS conditions.

▶ Similar deformed equations appear in the works of **Leung** by looking at vector bundles over a symplectic manifold and considering moment maps of various differential forms on the space of connections.

The deformed Hermitian-Yang-Mills equation

- **Existence and uniqueness** of solutions to the deformed Hermitian-Yang-Mills equation was extensively studied by **Collins-Jacob-Yau** in the case that (X, α) is an arbitrary Kähler manifold and $[\omega]$ is any class in $H^{1,1}(X, \mathbf{R})$.

- We **prove existence for many important cases assuming some form of stability for the** $(1,1)$**-class**.

- The necessary stability bears a strong formal relationship to slope stability, and hopefully it can be related to the program of **Thomas-Yau** on stability of Lagrangian submanifolds and mean curvature flow.

- Furthermore, the space of stable classes on (X, α) gives a good candidate for the mirror of the moduli space of polarized complex structures on the mirror manifold.

SYZ transformation of coisotropic A-branes

- **Kapustin and Orlov** speculated that the Fukaya category, which consists of Lagrangian branes, should be enlarged in order to capture the geometry away from the large volume limit. They suggested that extra objects called **coisotropic branes** are needed to be added into the category of A-branes.

- A coisotropic A-brane consists of a coisotropic submanifold (defined by $T^{\perp_\omega} X \subset TX$) and a unitary line bundle satisfying certain compatibility conditions.

- **Leung-Zhang** constructed the transform for coisotropic A-branes in the semi-flat case by using a **fiberwise Nahm transform**, which is a generalized version of the fiberwise Fourier-Mukai transform.

Fiberwise Nahm transform
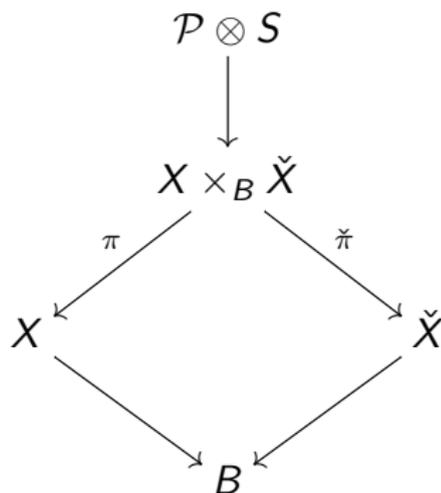
$$\mathcal{F}^{\mathrm{SYZ}}(-) := \check{\pi}_*(\pi^*(-)\otimes\mathcal{P}\otimes S).$$

- The new ingredient is S, which is a fiberwise spinor bundle, namely $S|_{\{\check{x}\}\times_B X}$ is the spin bundle over $\{\check{x}\}\times_B X \cong T$ (with respect to the flat metric on T induced from an affine metric on B).

- The push forward $\check{\pi}_*$ is defined by taking the kernel of the fiberwise Dirac operator $D_{\check{x}}$ of the spinor bundle.

- **Leung-Zhang** shows that semi-critical complex branes are transformed to semi-affine coisotropic branes, and vice versa.

SYZ and the mirror map

- The **mirror map** is a central object in mirror symmetry. It provides a canonical local isomorphism between the Kähler moduli and the mirror complex moduli around the large volume limit. The success of mirror symmetry on counting rational curves in the quintic threefold essentially relies on identifying the mirror map.

- The SYZ mirror is a family of CY varieties over the Kähler moduli. Thus it produces a canonical map from the Kähler moduli to the mirror complex moduli around the large volume limit.

- The **SYZ map** is written in terms of **open Gromov-Witten invariants**, which serve as quantum corrections for the SYZ construction. The invariants are **counting holomorphic discs**, which are difficult to compute, and hence in general it is difficult to write down the SYZ map explicitly.

SYZ and the mirror map

- **Gross-Siebert** formulated a general conjecture on the relation between mirror maps and the SYZ construction for toric degenerations. Namely, they conjectured that the degeneration family reconstructed from their program is canonically written in terms of flat coordinates.

- For toric Calabi-Yau manifolds, **Chan-Cho-Lau-Tseng** proved that mirror maps equal to the SYZ maps. Since the mirror map has explicit expressions in terms of hypergeometric series by solving Picard-Fuchs equations, **the equality provides a closed formula of open GW invariants** and an enumerative meaning of the mirror map.

From SYZ to closed-string mirror symmetry

- The SYZ map $\mathbf{f}^{\mathrm{SYZ}}$ should give a local identification between the Kähler moduli and the mirror complex moduli.

- In principle, closed-string mirror symmetry should follow from taking the derivative of the SYZ map $\mathbf{f}^{\mathrm{SYZ}}$. Namely, $\mathbf{f}^{\mathrm{SYZ}}$ should give a **local isomorphism of Frobenius structures**. In particular, the tangent map of $\mathbf{f}^{\mathrm{SYZ}}$ gives an isomorphism of the tangent spaces, which are quantum cohomology and variation of Hodge structures respectively.

- It was realized by **Fukaya-Oh-Ohta-Ono** for each compact toric manifold X, whose SYZ mirror is a Landau-Ginzburg model W. Its derivatives $\partial_q W$ in Kähler parameters q give an isomorphism $\mathrm{QH}(X) \cong \mathrm{Jac}(W)$.

- Closed-string mirror symmetry can also be deduced from taking the Hochschild homology of $D\mathrm{Fuk}(X) \cong D\mathrm{Coh}(\check{X})$. This direction is currently studied intensively by **Ganatra-Perutz-Sheridan**. All of these constructions should combine harmonically to give a complete picture.

Twin Lagrangian fibrations v.s. elliptic fibrations

- Elliptic Calabi-Yau manifolds play important roles in string theory and F-theory.
- Fourier-Mukai transform between dual elliptic fibrations corresponds to spectral cover construction.
- **Leung**-**Yau** proposed that the SYZ mirrors to elliptic Calabi-Yau manifolds admit twin Lagrangian fibrations. Moreover **Fourier**-**Mukai transform should be mirror to Lagrangian correspondence**.
- This research direction is further studied by **Leung**-**Li**. They constructed twin Lagrangian fibrations on toric Calabi-Yau manifolds and affine conic bundles. Using family Floer theory, they showed that these Lagrangian fibrations are mirror to holomorphic (but not necessarily elliptic) fibrations on the mirror.

Section 3

Future perspectives

SYZ in the presence of flux

- In string theory, when Ramond-Ramond flux is present, the spacetime is modeled by **non-Kähler Calabi-Yau threefolds**. The Type IIA and IIB geometries are defined by the following equations. ρ_A and ρ_B are the sources of the flux.

Type IIA:

$$\mathrm{d}\omega = 0,$$

$$\mathrm{d}(\mathrm{Re}\,\Omega) = 0,$$

$$\Omega \wedge \bar{\Omega} = -\mathbf{i}\,F \cdot \frac{\omega^3}{6},$$

$$\mathrm{dd}^\Lambda(F \cdot \mathrm{Im}\,\Omega) = \rho_A.$$

Type IIB:

$$\mathrm{d}\Omega = 0,$$

$$\mathrm{d}(\omega^2) = 0,$$

$$\Omega \wedge \bar{\Omega} = -\mathbf{i}\,F \cdot \frac{\omega^3}{6},$$

$$2\mathbf{i}\partial\bar{\partial}\left(F^{-1} \cdot \omega\right) = \rho_B.$$

- **Lau-Tseng-Yau** showed that these two systems are SYZ mirror to each other in the semi-flat setting. It is a very interesting question how to extend mirror symmetry beyond the semi-flat context in the presence of flux.

- When H-flux is present, the spacetime is modeled using **generalized complex geometry**. T-duality was studied by **Cavalcanti-Gualtieri** and several physics groups. SYZ mirror symmetry in this context is not understood yet.

Generalizing SYZ by non-tori

- **Aganagic-Vafa** constructed non-compact Lagrangian branes, which has the topology of $\mathbf{R}^2 \times \mathbf{S}^1$, to construct generalized mirrors of the resolved conifold, which has important applications in knot theory. It indicates that Lagrangians which are not necessarily tori can be useful for mirror construction.

- **Seidel and Sheridan** used Lagrangian immersions to prove homological mirror symmetry for the genus-two curves and Fermat-type Calabi-Yau hypersurfaces. It is an interesting question **how to use Lagrangian non-tori to construct the mirror**.

- **Cho-Hong-Lau** is formulating a construction scheme using infinitesimal obstruction theory of general Lagrangian branes. It always comes with a canonical **functor** which transforms Lagrangians to (twisted) coherent sheaves.

SYZ in the global moduli

- ▶ The SYZ map $\mathbf{f}^{\mathrm{SYZ}}$ is believed to be a local piece of a global mirror isomorphism between the moduli spaces. Thus the SYZ mirrors should extend over the global moduli.

- ▶ For elliptic orbifolds, **Lau-Zhou** proved that their SYZ mirrors are expressed in terms of **modular forms**. In particular they can be extended to the global moduli (with non-trivial monodromy around the limit points). The same technique can be applied to rigid Calabi-Yau manifolds to deduce the global modular properties of their SYZ mirrors.

- ▶ For A_1 singularities and their fiber products, an ongoing work of **Kanazawa**-**Lau** constructed their SYZ mirrors and showed that they are indeed modular objects. In particular, they deduce the 24th-root of Yau-Zaslow formula, which expresses open Gromov-Witten invariants in terms of the Dedekind η function.

- ▶ The works suggest a **geometric program to construct modular forms** using SYZ which have important applications in number theory.